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The Optimal Relationship Between the Fundamental Frequency of Drums and the Random Exit Time of Particles

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The question of whether or not the ear can detect size and shape of a drum has intrigued mathematicians for decades. The question was partially answered in 1992 by Gordon, Webb, and Wolpert who proved the existence of two drumheads of different shapes that have the same infinite sequence of frequencies [1]. Although the ear is not reliable to hear the shape of a drum, it is possible to hear the difference in the size of a drum because different size drums create different frequencies when struck. However, frequencies are also determined by where the drum is struck and by the shape and size of a string.

A string in the one dimension maintains a relatively simple motion as frequency increases. The frequency or pitch of a string is also related to the shape or size of the string. A shorter string has high frequency while a very long string would have a low frequency. Moving up to the second dimension, one can imagine the motion of a planar object such as the motion of a drumhead. Similarly, as the case of a string, the shape and size of a drumhead affects its frequency. The Wave Equation describes the motion of a drumhead, which is given by the partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

where c is some physical constant related to the drum. From this equation, one can obtain the eigenvalues known as the fundamental frequency $\lambda_1(D)$.

Thus, drum frequencies are determined by a number of factors, including where the drum is struck and frequencies can vary because of Brownian motion or random movement of particles. Brownian motion was discovered upon observation of pollen by the botanist Robert Brown in 1827. He noticed that the motion of pollen grains moved in a random and jittery motion. In 1905, Albert Einstein used Brownian Motion to confirm the existence of atoms and molecules. It has also been used to explain the random and jittery change in stock prices. This random motion has been studied extensively in mathematics and is now known as Brownian motion. In the case of a drumhead, or more precisely a set $D \subset \mathbb{R}^2$ in the plane, one must study the behavior of Brownian motion in the drumhead D until it exits the drum [2]. The average exit time of the Brownian motion from the drum D started at the point x is denoted here by $\mathbb{E}[\tau_D]$. Here, the Brownian motion process gets killed when the random particle reaches the boundary.

There exists an inverse relationship between bigger and smaller drums. The function we are trying to optimize is,

$$G(D) = \lambda_1(D) \cdot \max_{(x,y)\in D} \mathbb{E}\left[\tau_D\right],$$

where $\lambda_1(D)$ is the fundamental frequency and $\max_{(x,y)\in D} \mathbb{E}\left[\tau_D\right]$ is the maximum expected exit time from the drum *D*. For larger drums, the fundamental frequency is low while the average exit time of the particle is large. This makes sense as big drums have low sounds, but since the drum is big it would take longer for the particle to exit. While for smaller drums, the fundamental frequency is high and the average exit time is small. Hence the function *G* can be optimized by analyzing and calculating the optimal relationship between the fundamental frequency and average exit time.

Where *D* represents any drum, it is not possible to optimize the function *G* over all drums given the current knowledge in the fields of Mathematics and Physics. The variety in size and shape over all possible drums leaves too many variables to consider. For that reason, any research done on the function *G* typically focuses on one shape. The smaller, more achievable goal of optimizing the function for one drumhead shape can eventually lead to the greater goal of optimizing the function over all drumhead shapes. The research done here focuses on the drumhead shape of Ellipses.

The functional *G* has been studied extensively in the literature. A study found that there exists a constant $C \geq 0$ such that

$$2 \leq G(D) \leq C$$

in the case of simply connected domains [2]. Later, two papers proved independently that the bounds hold true for general domains in $\mathbb{R}^n$ [3][4].

The main problem in this research is to find the optimal relationship for drums whose shapes are Ellipses. An elliptical drum is defined by the set

$$E_{a,b} = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\},$$

where *a* and *b* are the major and minor radius, respectively. The following Theorem was obtained for Elliptical drums.

Theorem 1. *For any $a, b > 0$, we have*

$$G(E_{a,b}) = \lambda_1(E_{a,b}) \cdot \max_{(x,y)\in E_{a,b}} \mathbb{E}\left[\tau_{E_{a,b}}\right] \leq \sqrt{2} + \frac{3}{2}.$$

This theorem was obtained by computing $\max_{(x,y)\in E_{a,b}} \mathbb{E}\left[\tau_{E_{a,b}}\right]$ exactly by solving the related differential equation. While the fundamental frequency $\lambda_1(E_{a,b})$ was estimated by using a bound on its variational formulation. In future research, we hope to obtain sharper bounds for ellipses and explore other shapes.

These findings serve as a stepping stone that brings mathematicians and physicists one step closer to being able to generalize the optimal ratio between the fundamental frequency and expected exit time across all drum shapes and sizes. We calculated $G(E_{a,b})$, the optimization of $G(D)$ specifically for ellipses, which is just one piece of the optimization puzzle for the function $G(D)$ for all drums, D.

References

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